1.4 Indefinite equilibrium equations

Constitutives equations and strain-displacement relations are not enough to solve an elastic problem: equations (1.3) give the tensional status once the strains are known, while equations (1.4) give the strain status as a function of the displacement of the point where the analysis is being carried out.

Therefore some relations are missing, in particular some equations that relate the origin (forces that load the structure) to the effect (displacement, deformations, stresses)

Let **f** be a volume force (i.e. weight, inertial forces) defined by its components f_x, f_y, f_z in the chosen coordinate system; the variation of the stress, that can be recorded by moving between two parallel faces of a small cube extracted in the neighbourhoods of point P, is in equilibrium with volume forces (see figure 3). By writing translational equilibrium equations in the three directions x, y and z we have:

$$\begin{aligned} &\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z}+f_x=0 \\ &\frac{\partial \sigma_{yy}}{\partial y}+\frac{\partial \tau_{yx}}{\partial x}+\frac{\partial \tau_{yz}}{\partial z}+f_y=0 \\ &\frac{\partial \sigma_{zz}}{\partial z}+\frac{\partial \tau_{zy}}{\partial y}+\frac{\partial \tau_{zx}}{\partial x}+f_z=0 \end{aligned} \qquad (1.7)$$

In matrix notation equations (1.7) will be:

$$[D]^T \cdot \{\sigma\}+\{f\}=0 \qquad (1.8)$$

And, by writing in the extended form equations (1.8):

$$\begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial z} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \cdot \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{zy} \\ \tau_{zx} \end{Bmatrix} + \begin{Bmatrix} f_x \\ f_y \\ f_z \end{Bmatrix} = 0$$

Summarising, we have a system constituted by three matricial equations:

$$\{\sigma\}=[E]\cdot\{\varepsilon\} \qquad (1.3)$$

$$\{\varepsilon\}=[D]\cdot\{u\} \qquad (1.5)$$

$$[D]^T\cdot\{\sigma\}+\{f\}=0 \qquad (1.8)$$

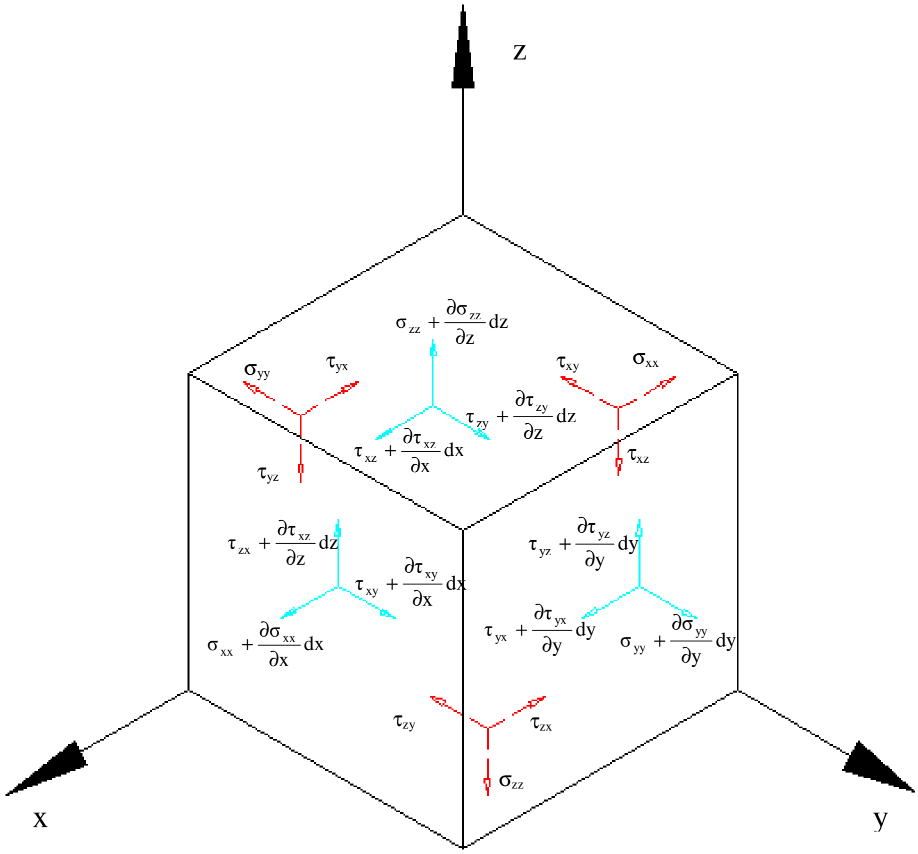

Figure 3. Stresses acting on the faces of a small cube extracted from the neighbourhood of point P. In the figure, for clarity, volume forces are not present.

This system is constituted by 6 algebraic equations (1.3) and by 9 linear partial differential equations (1.5 and 1.8); also the unknowns are 15: the 3 displacement components, the 6 strain components, and the 6 stress components. This system can be solved, but it presents a lot of analytical difficulties.