B.3 Shape functions for a plane stress triangular element

Shape functions are polynomial equations that describe the displacement field of any point inside the element as a function of the displacements that the element nodes undergo.

The polynomial degree depends on the node number (in fact beside the nodes at the vertices it is possible to have elements with nodes along its sides). The higher the polynomial degree the better the element behaviour will generally be, as it will be clearer in what follows.

Let us consider a point P belonging to a triangular element (see figure 1).

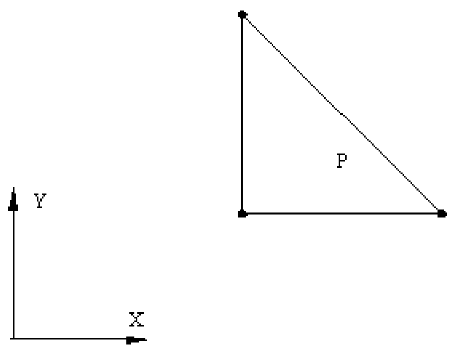

Figure 1. Point P belonging to a plane stress triangular element.

For the given point P we can write (see also § 6.3):

$$u_x(P) = a_1 + a_2x + a_3y$$
$$u_y(P) = a_4 + a_5x + a_6y \qquad \text{(B.2)}$$

where x and y are the point P coordinates and a_i (i = 1....6) are some constants to be determined in the following way.

If we suppose to know the displacement components of the element nodes, due to equations (B.2) we will have:

$$u_{1x} = a_1 + a_2x_1 + a_3y_1$$
$$u_{1y} = a_4 + a_5x_1 + a_6y_1$$
$$u_{2x} = a_1 + a_2x_2 + a_3y_2$$
$$u_{2y} = a_4 + a_5x_2 + a_6y_2$$
$$u_{3x} = a_1 + a_2x_3 + a_3y_3$$
$$u_{3y} = a_4 + a_5x_3 + a_6y_3$$

where u_{jx} and u_{jy} (j = 1...3) are the displacement components of the j node, x_j, y_j are the node j coordinates.

In matricial form we can write:

$$\{u_{nod}\} = \begin{bmatrix} 1 & x_1 & y_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_1 & y_1 \\ 1 & x_2 & y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_2 & y_2 \\ 1 & x_3 & y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_3 & y_3 \end{bmatrix} \cdot \{a\}$$

$$\{u_{nod}\} = [N] \cdot \{a\}$$

from which we retrieve:

$$\{a\} = [N]^{-1} \cdot \{u_{nod}\} = [\Phi] \cdot \{u_{nod}\}$$

By writing the [Φ] matrix in such a way to point out its rows we will have:

$$\{a\} = \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \\ \Phi_5 \\ \Phi_6 \end{bmatrix} \cdot \{u_{nod}\} \qquad \text{(B.3)}$$

By substituting equations (B.2) in equations (1.11) (see Chapter 1) and by executing the derivatives we can obtain the strain tensor components:

$$\varepsilon_x = a_2$$
$$\varepsilon_y = a_6$$
$$\gamma_{xy} = a_3 + a_5$$

As a_i are constant it is clear why this kind of element is called Constant Strain Triangle (CST). Therefore, in order to adequately catch the deformation field in a structure modelled with this element it is necessary to use a great number of them. Elements with polynomial that contain also quadratic forms (such as the 6 noded triangle) will see a deformation field which varies inside the element, thus giving a higher sensitivity to stress gradients.

As, due to equations (B.3), we have:

$$\{a_2\} = \{\Phi_2\} \cdot \{u_{nod}\}$$
$$\{a_3\} = \{\Phi_3\} \cdot \{u_{nod}\}$$
$$\{a_5\} = \{\Phi_5\} \cdot \{u_{nod}\}$$
$$\{a_6\} = \{\Phi_6\} \cdot \{u_{nod}\}$$

we can write:

$$\varepsilon_x = \{\Phi_2\} \cdot \{u_{nod}\}$$
$$\varepsilon_y = \{\Phi_6\} \cdot \{u_{nod}\}$$
$$\gamma_{xy} = \{\Phi_3 + \Phi_5\} \cdot \{u_{nod}\}$$

or, in a more compact form:

$$\{\varepsilon\} = \begin{bmatrix} \Phi_2 \\ \Phi_6 \\ \Phi_3 + \Phi_5 \end{bmatrix} \cdot \{u_{nod}\} = [B] \cdot \{u_{nod}\} \qquad \text{(B.4)}$$

B.4 The stiffness matrix for the CST element

In order to retrieve the stiffness matrix for the CST element we will use an energy based method, starting from a simple example.

The external work necessary to move an end of spring with k stiffness (being the other end constrained) by the quantity x is:

$$L_e = \int_0^x k \cdot x \cdot dx = \frac{1}{2} \cdot k \cdot x^2$$

In a similar way, being $[K_e]$ the matrix (still to be defined) of an element, the work of the external forces required to impose the nodal displacements $\{u_{nod}\}$ will be:

$$L_e = \frac{1}{2} \cdot \{u_{nod}\}^T \cdot [K_e] \cdot \{u_{nod}\}$$

while the internal forces work is given by the following relationship: