membrane component from "arising", since there is no constraint that can counteract it. With a linear calculation this effect is obviously lost.

By means of a simple example we have highlighted an important aspect of the automatic calculation of structures: in certain situations the possibility of taking into account geometric nonlinearities can make valid a structural project that the linear calculation would have discarded, while in other circumstances there is no practical benefit in using nonlinear calculation techniques, more refined and expensive in terms of hardware resources engagement.

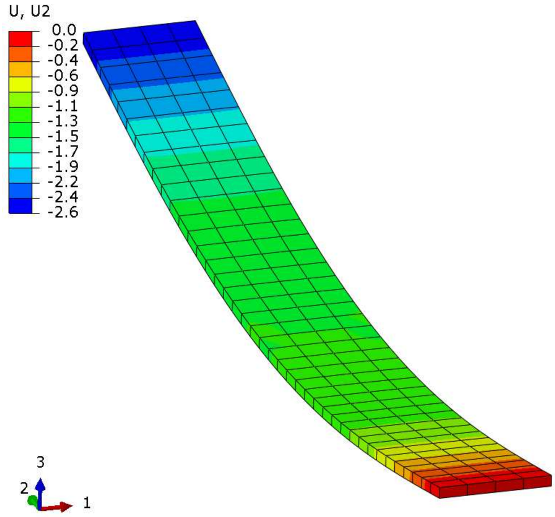

Figure 11.16. Nonlinear calculation with hinge-simple support constraints: displacement in the longitudinal direction.

The experience of the structural engineer is needed to determine which way to proceed in the calculation of a structure. Finally, if a geometric nonlinear solution is required, it may be convenient to use substructuring (see Chapter 7). Many codes on the market today are able to manage superelements in a totally transparent way to the user, creating them within a nonlinear analysis. If, for example, we have a structure which, due to its geometric conformation and the load conditions to which it is subjected, sees only one or more of its parts undergoing large displacements, we can think of inserting the part undergoing small displacements in a single internal superelement, leaving only the nonlinear part or parts on the outside. In this way the calculation code, once assembled the stiffness matrix of the superelement (linear) does not "touch" it anymore during the iterations, which then will be faster as they are performed on a smaller number of degrees of freedom. It is the calculation code itself, once it has reached convergence, to "expand" the results to the superelement. The only difficulty for the user is to decide which part will be enclosed in the superelement and which will remain outside.

11.4 Post-buckling

At this point we are able to analyze situations that may lead structures to work beyond the load of elastic buckling (post-buckling). This type of condition is usually improperly defined "non-linear buckling", thus implying that a nonlinear analysis with large displacements is required to investigate this phenomenon. In fact, the phenomenon is generally associated with geometric nonlinearity, but it is not excluded that there may also be plasticization of one or more zones and/or contact with other parts of the structure.

11.4.1 Beam in compression

To introduce this particular structural problem we will again use the beam used in the previous example. The geometric characteristics are those of figure 11.1, while the constraint conditions see a hinge at one end and a simple support at the opposite end. In this way the condition shown in figure 11.17 is achieved.

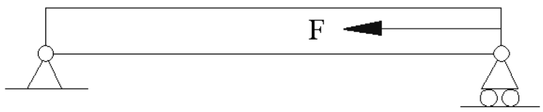

Figure 11.17. Constraint and load conditions for the beam.

Under these assumptions, the critical buckling load is (see Chapter 5):

$$P_{cr} = \frac{\pi^2 \cdot E \cdot I_x}{L^2} = 678 \text{ N}$$

The finite element calculation, carried out according to the methods indicated in Chapter 5 for linear buckling, gives a value practically identical to the one calculated above and therefore we will not focus further on the linear aspect. In order to perform a post buckling analysis, it is first necessary to perform a calculation step to "suggest" to the code what deformation the structure will tend to assume under load.

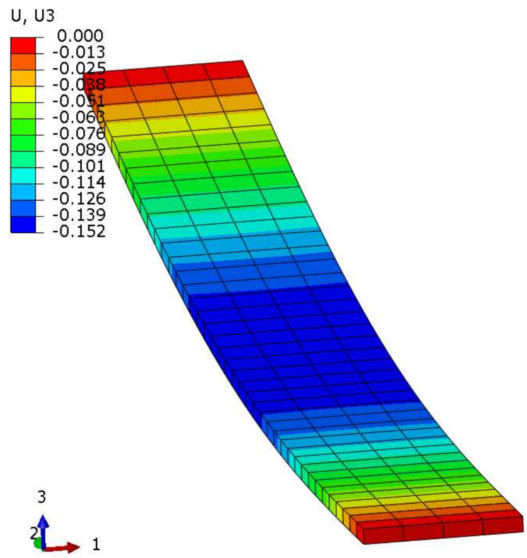

Figure 11.18. "Predeformed" beam. The vertical component of the displacement is plotted here (amplification factor = 50).

In the case of our beam it will be sufficient to apply, for example, a very small pressure value to make the beam slightly deflect (in cases where it is not easy to establish the shape of the buckling mode it is sufficient to perform a linear elastic buckling calculation). This is done because otherwise the phenomenon cannot occur numerically, as the model of the structure is "perfect" and could never reach collapse. A second precaution is to apply not a force, but a displacement; this is because the structure, once unstabilized, cannot react to a load greater than the critical one. If for example in our case we applied a force $F > P_{cr}$ the structure would be labile and the solution could not reach convergence. Vice versa by imposing a "controlled" displacement this does not happen. However the discourse will be clearer once the proposed example is studied. Figure 11.18 contains the beam " predeformed" by a uniform pressure p = 0.001 MPa. As it can be observed the deflection at the center of the span is about 0.15 mm.

Following the application of the pressure load, an axial displacement of 1 mm is imposed on the simply supported end such that the beam is compressed. The deformation of figure 11.19 is obtained. Now we can observe a deflection at the center of the span of almost 9 mm. Clearly the axial displacement of the simply supported end will be equal to 1 mm. From these data we are still not able to understand if the beam has buckled or not.

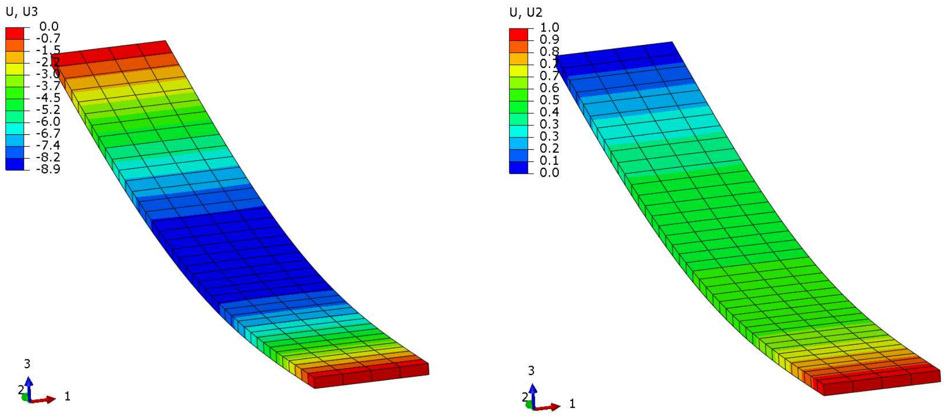

Figure 11.19. Left: displacements in the z-direction; right: displacement in the longitudinal direction (amplification factor = 1.0).

We can say that, in a linear calculation hypothesis, imposing a displacement of 1 mm on the beam generates a deformation $\varepsilon = 1/200 = 0.005$ to which corresponds a stress $\sigma = \varepsilon \cdot E = 1030$ MPa that would ultimately give a force $F = \sigma \cdot A = 41200$ N that is well beyond the critical load of the bar. But what happened in reality? Let's try to graph the reaction force in the axial direction as a function of the displacement imposed on the simply supported end. We obtain the curve of figure 11.20.

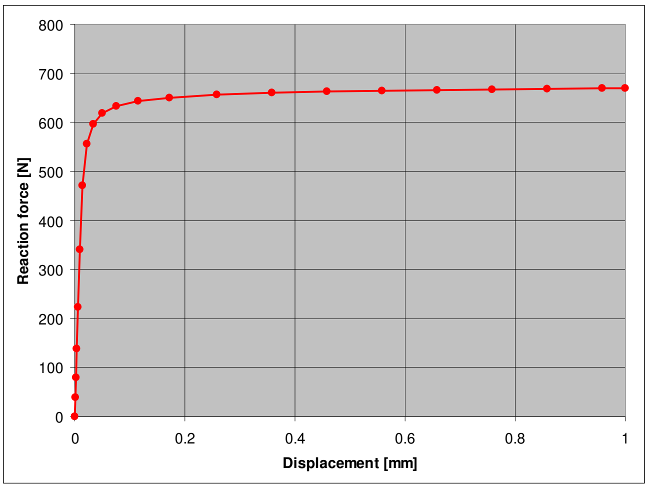

Figure 11.20. Reaction force in the axial direction plotted as a function of the displacement imposed on the simply supported end. An asymptote is clearly distinguishable.

The value of the reaction force at the end of the loading step is 669.5 N, very close to that calculated with Euler's formula. It is not surprising that this number is lower: we have in fact "predeformed" the beam to simulate a straightness defect. If we were to perform a linear buckling calculation on the already deformed beam, the value determined in the beginning would also tend to decrease.However, we must observe that the curve begins to deviate from linearity at a much lower value, i.e. already around

550 N: we can therefore state that the linear calculation produces non-conservative results and that, therefore, if we do not apply a post-buckling analysis, it is better to keep high safety coefficients (at least equal to 1.5, unless we then resort to physical tests).

Finally, figure 11.21 contains the stress contour in the longitudinal direction. Clearly there is a bending stress, preponderant, in addition to the membrane component:

$$\sigma_m = \frac{\sigma_{top} + \sigma_{bot}}{2} = -16.6 \text{ MPa} \qquad \sigma_b = \frac{\sigma_{top} - \sigma_{bot}}{2} = \pm 458 \text{ MPa}$$

Obviously, the σ_m we obtained from figure 11.21 is equal to the reaction force (669.5 N) divided by the area of the section (40 mm^2).

In any case, the stress is definitely lower than that, purely membra-nal, which would occur if the beam did not become unstable (and equal to 1030 MPa, as seen).

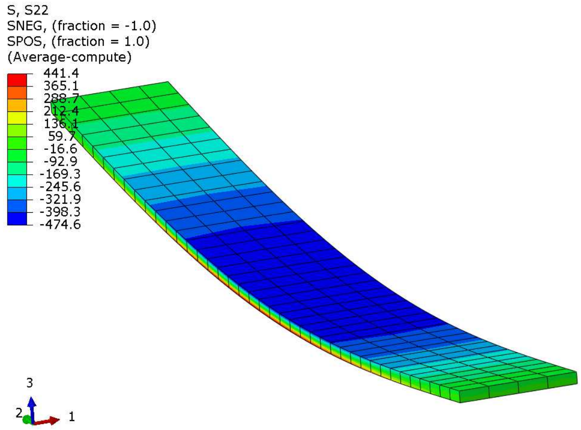

Figure 11.21. Stress in the longitudinal direction. There is a significant bending component against a modest membrane one.

We then observe that if the material works below its yield value, removing the load would return the structure to its original condition.

At this point it is clear why in a calculation of this kind it is not possible to apply directly the force to which the structure is supposed to resist: we have in fact seen that the beam was able to react only up to a certain value of the force F, even if at its extreme the imposed displacement continued to increase. A force "out of control" would not have led to the convergence of the solution. And this happens because the applied load, since the structure has become unstable, has no way to "flow" through other parts. We recall that a labile structure has a non positive-defined stiffness matrix and therefore, in practice, the application of a force greater than the critical load is assimilated to attempting to solve an unconstrained structure. The imposition of a displacement, on the contrary, implies the application of a constraint (in particular a constraint on the degree of freedom become labile) and this results (as reported in Appendix B) in the elimination of the rows and columns of the stiffness matrix corresponding to the degrees of freedom constrained (in reality many software use the "penalty method", which consists in attributing to the constrained degrees of freedom very high stiffnesses instead of eliminating the corresponding rows and columns; by doing so there is the advantage of not having to partition the matrix for the calculation of the reaction forces, but the displacement of the constrained degrees of freedom can never be "numerically" zero). This is the reason why with the application of a load the solution of the beam in compression does not converge while imposing the displacement to the simply supported end the calculation has a positive outcome. If we had a structure consisting of several components and one of them went into buckling it would not be said that the structure as a whole would collapse, as we will see in the next example, where, for this reason, even with the application of a force the calculation converges.

11.4.2 Planar frame

Let's suppose we have the planar frame shown in figure 11.22 through the related beam-type finite element model. From a first linear buckling analysis we find that F_{cr} = 10512 N. Clearly the element that becomes unstable is the diagonal A, as we can see from figure 11.23. We want to see what happens to the structure when we apply an almost double force F = 20000 N.

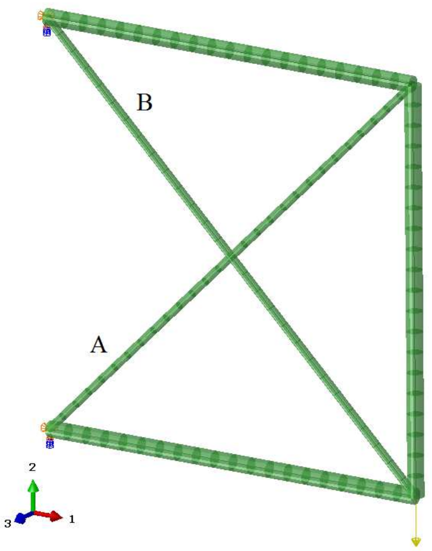

Figure 11.22. Planar steel frame. The sides of the square are 500 mm long; the diameter of the beams placed along the sides is 20 mm while the diameter of the two diagonals is 10 mm. The beam elements are shown with their actual geometry.

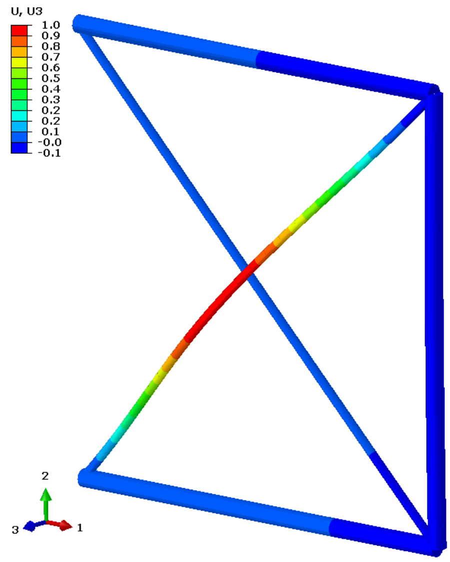

Figure 11.23. Linear calculation results: the first buckling mode says that the load that instabilizes the diagonal A is 10512 N (amplification factor = 50).

We expect that the diagonal A, which certainly works in compression, will at some point become unstable, as predicted by the linear analysis. However, the diagonal B, which instead works in traction, will not buckle and therefore the structure as a whole may not even collapse, despite having one element buckled.

Also in this case it is necessary to "suggest" to the code that the diagonal A will buckle; we will do this by applying a modest force (for example 10 N) in the central point of the diagonal itself and directed perpendicularly to the plane of the truss, that is along z (the direction according to which the buckling will occur is obtained from the