APPENDIX B
The stiffness matrix for the plane stress 3-node element

B.1 Introduction

As already mentioned in other parts of the text, the result of an linear elastic structural problem obtained with FEM, once the structure to be calculated has been defined (geometry, materials, internal constraints, boundary conditions), is nothing but the solution of the following matrix equation:

$$\{u\} = [K]^{-1} \cdot \{F\} \qquad \text{(B.1)}$$

being {u} the vector of (unknown) displacements of predefined points, the nodes, of the structure, {F} the vector of forces applied at some (or even all) nodes, [K] the so-called global stiffness matrix of the structure itself.

B.2 Finite Elements

As mentioned in Chapter 1, finite elements are domains in space (from 1 to 3 dimensions depending on the type of element) within which the solution of the elastic problem is approximated. It is therefore clear that the smaller (and therefore the more numerous) the elements that model the structure, the better the accuracy of the solution; at the theoretical limit, an infinite number of finite elements guarantees the exact result. It is therefore the experience of the structural engineer that intervenes in establishing which "density" of elements is suitable to provide an engineering-correct value for a given problem. In order to solve (B.1), it is necessary to know the matrix [K] which, as we have said, is a function of the geometry and the material that makes up the structure. The matrix [K] is constructed by the calculation code by assembling in an appropriate way the various matrices $[K_e]$ of the single elements in which the entire structure has been subdivided.

In the following paragraphs we will see how to determine $[K_e]$ for a 3-node triangular element in a plane stress state.

B.3 Shape functions for the plane stress triangular element

Shape functions are polynomials that describe the displacement domain of points within the element in relation to the displacements that the nodes of the element undergo.

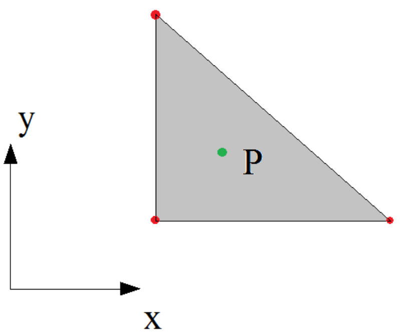

Figure B.1. Point P belonging to a triangular element in a plane stress state.

The degree of this polynomial depends on the number of nodes (in addition to the nodes at the vertices, there can be elements with nodes along the sides). The higher the degree of the polynomial, the better the behavior of the element, as it will be clearer in the following.

Consider a point P belonging to a triangular element with 3 nodes (see figure B.1).

For the point P in question we may write (see also § 7.3):

$$u_x(P) = a_1 + a_2x + a_3y$$
$$u_y(P) = a_4 + a_5x + a_6y \qquad \text{(B.2)}$$

where x and y are the coordinates of point P and a_i (i = 1....6) are constants to be determined in the following way.

If we assume that we know the components of the displacement of the nodes of the element, by virtue of (B.2), we will have:

$$\begin{aligned} u_{1x} &= a_1 + a_2x_1 + a_3y_1 \\ u_{1y} &= a_4 + a_5x_1 + a_6y_1 \\ u_{2x} &= a_1 + a_2x_2 + a_3y_2 \\ u_{2y} &= a_4 + a_5x_2 + a_6y_2 \\ u_{3x} &= a_1 + a_2x_3 + a_3y_3 \\ u_{3y} &= a_4 + a_5x_3 + a_6y_3 \end{aligned}$$

where u_{jx} and u_{jy} (j = 1...3) are the displacement components of node j, x_j, y_j are the coordinates of node j.

In matrix form we will write:

$$\{u_{nod}\} = \begin{bmatrix} 1 & x_1 & y_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_1 & y_1 \\ 1 & x_2 & y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_2 & y_2 \\ 1 & x_3 & y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_3 & y_3 \end{bmatrix} \cdot \{a\}$$

$$\{u_{nod}\} = [N] \cdot \{a\}$$

from which we derive:

$$\{a\} = [N]^{-1} \cdot \{u_{nod}\} = [\Phi] \cdot \{u_{nod}\}$$

Writing the matrix [Φ] in a way that explicitly expresses the rows we will have:

$$\{a\} = \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \\ \Phi_5 \\ \Phi_6 \end{bmatrix} \cdot \{u_{nod}\} \qquad \text{(B.3)}$$

Substituting (B.2) into (A.11) (see Appendix A) and performing the derivatives we obtain the components of the strain tensor:

$$\varepsilon_x = a_2$$
$$\varepsilon_y = a_6$$
$$\gamma_{xy} = a_3 + a_5$$

Since the a_i are constants it remains clear why this type of element is called Constant Strain Triangle (CST). Therefore, in order to adequately capture the strain field in a structure modeled with this element, it is necessary to use a large number of them. Elements with polynomials also containing quadratic shapes (e.g., the 6-node triangle) would see a variable strain field within the element itself, providing better sensitivity to stress gradients.

Since, by virtue of (B.3), we have:

$$\{a_2\} = \{\Phi_2\} \cdot \{u_{nod}\}$$
$$\{a_3\} = \{\Phi_3\} \cdot \{u_{nod}\}$$
$$\{a_5\} = \{\Phi_5\} \cdot \{u_{nod}\}$$
$$\{a_6\} = \{\Phi_6\} \cdot \{u_{nod}\}$$

we can write:

$$\varepsilon_x = \{\Phi_2\} \cdot \{u_{nod}\}$$
$$\varepsilon_y = \{\Phi_6\} \cdot \{u_{nod}\}$$
$$\gamma_{xy} = \{\Phi_3 + \Phi_5\} \cdot \{u_{nod}\}$$

Or, in a more compact form:

$$\{\varepsilon\}=\begin{bmatrix}\Phi_2\\ \Phi_6\\ \Phi_3+\Phi_5\end{bmatrix}\cdot\{u_{nod}\}=[B]\cdot\{u_{nod}\} \qquad \text{(B.4)}$$

B.4 The stiffness matrix for the CST element

To determine the stiffness matrix of the element in discussion we will use an energy method, starting with a simple example.

The external work required to move the end of a spring of stiffness k (the other end being constrained) by the quantity x is equal to:

$$L_e=\int_0^x k\cdot x\cdot dx=\frac{1}{2}\cdot k\cdot x^2$$

Similarly, since $[K_e]$ is the matrix (yet to be defined) of an element, the work of the external forces required to impose the nodal displacements $\{u_{nod}\}$ will be:

$$L_e=\frac{1}{2}\cdot\{u_{nod}\}^T\cdot[K_e]\cdot\{u_{nod}\}$$

while the work of the internal forces is given by the following relationship:

$$L_i=\frac{1}{2}\cdot\int_V\{\sigma\}^T\cdot\{\varepsilon\}\cdot dV$$

But, due to (A.10) (see Appendix A):

$$L_i=\frac{1}{2}\cdot\int_V\{\varepsilon\}^T\cdot[E]^T\cdot\{\varepsilon\}\cdot dV$$

Finally, due to (B.4):

$$L_i=\frac{1}{2}\cdot\int_V\{u_{nod}\}^T\cdot[B]^T\cdot[E]^T\cdot[B]\cdot\{u_{nod}\}\cdot dV$$

It is easy to realize that none of the terms under the integral sign depend on the domain of integration and therefore the integral is extended to only the term dV; moreover, by the Principle of Virtual Works, it must be $L_e = L_i$.